On Improvement of Convergence Rate of Short Wave Propagation Finite Element Models

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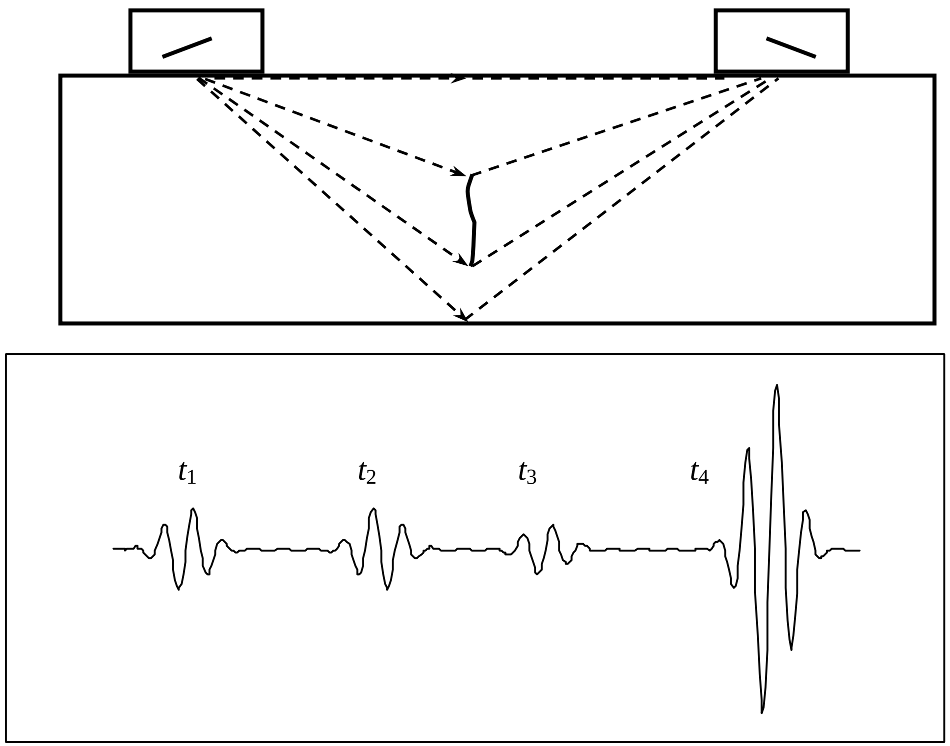
Short wave propagation models

- the length of the *short wave* is many times less than the dimensions of the structure

Applications in:

- Ultrasonic measurement process simulation;
- Seismic waves analysis;
- Water waves simulation;
- …

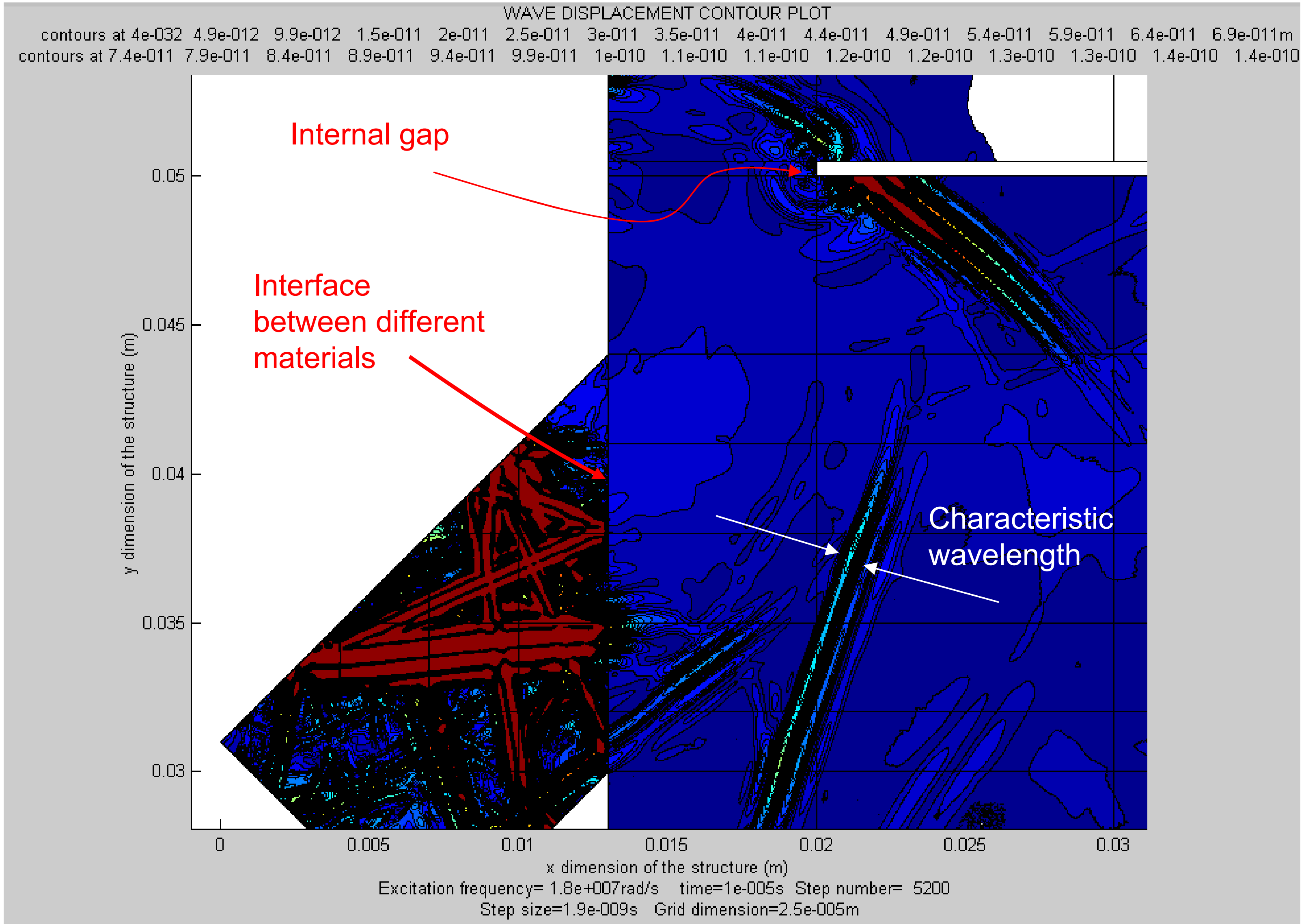
Application example:
Time-of-flight difraction (TOFD) method



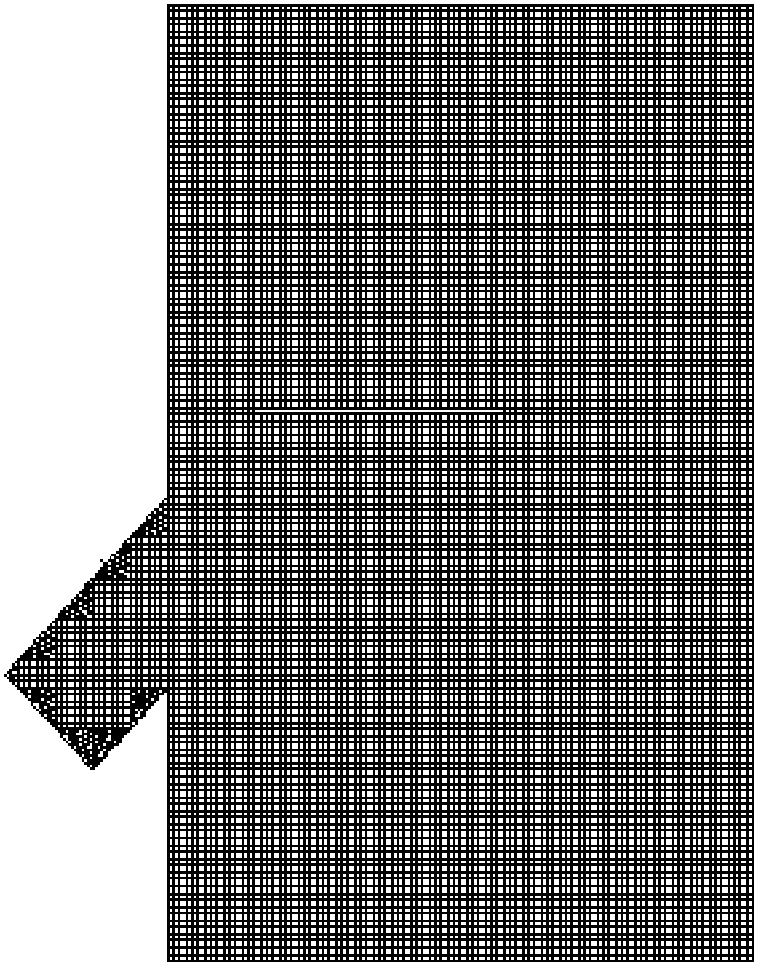
TOFD method geometry:

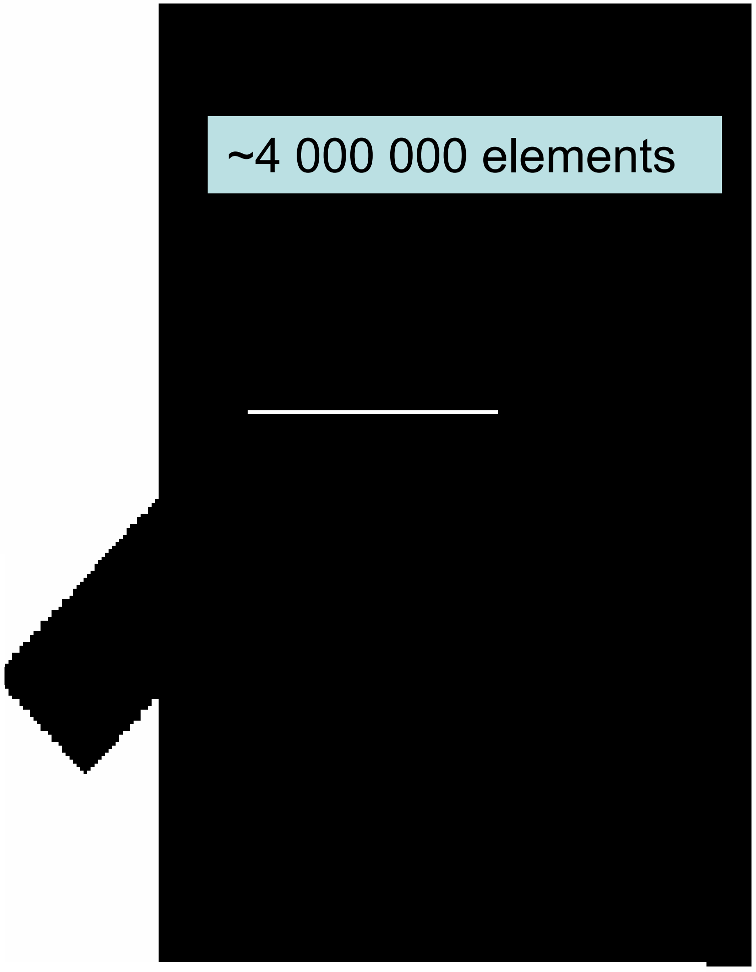
1. direct, subsurface wave;
2. wave diffracted from top point of the crack;
3. wave diffracted from bottom point of the crack;
4. signal reflected from the bottom of the plate

2D Example: Ultrasonic wave packages propagating in plexyglass and steel structure



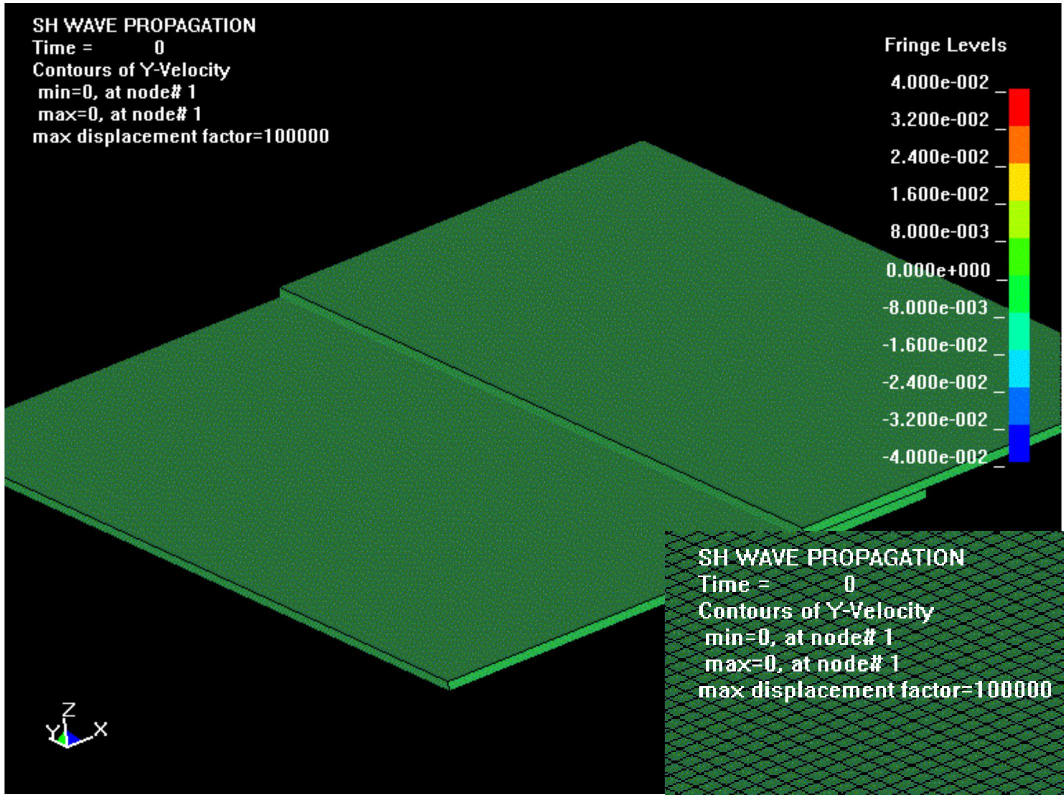
Modeling example : the mesh

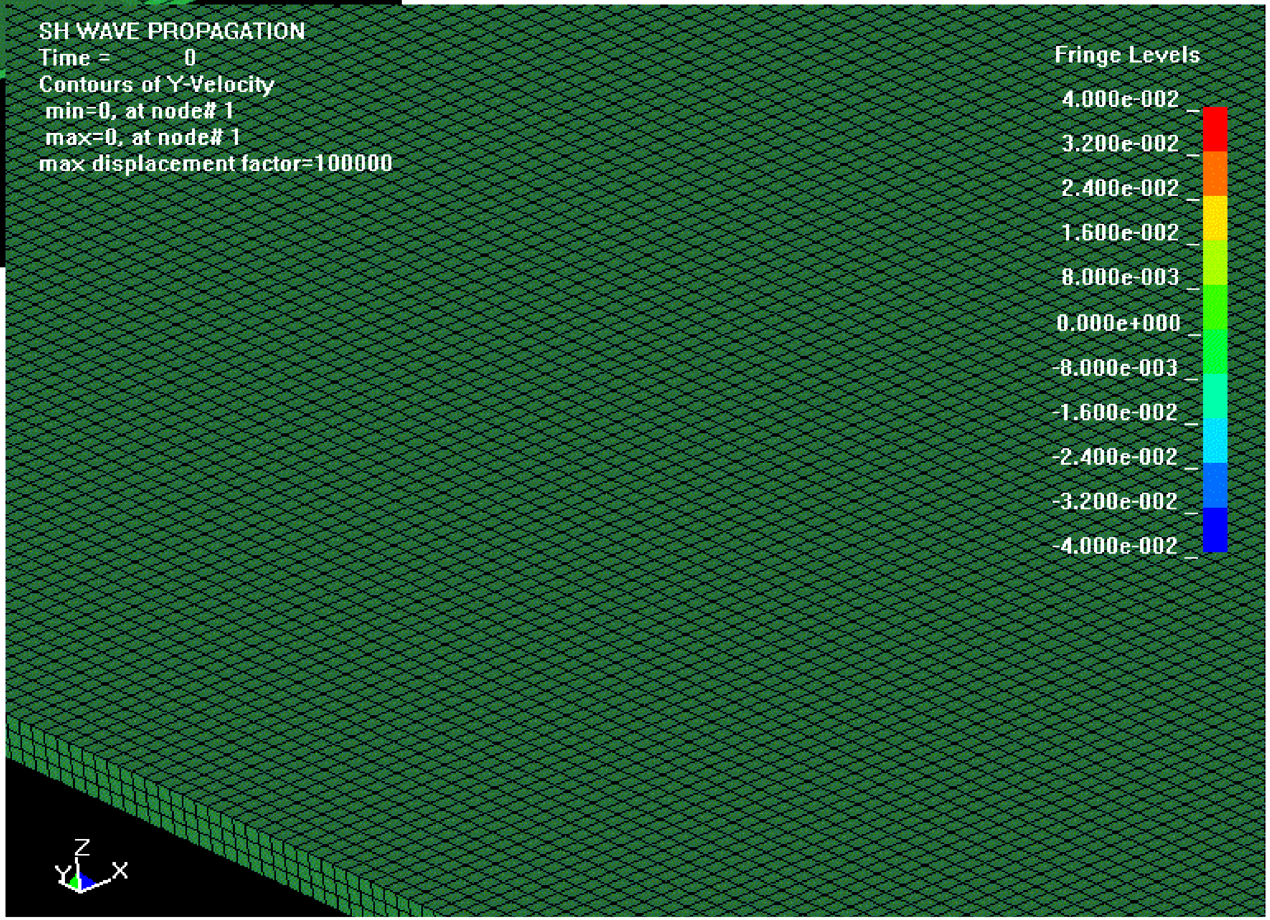




3D Example:

Ultrasonic wave propagation in 2 welded plates (simulation in LSDYNA)





Dynamic equation

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\}+[\mathbf{C}]\{\dot{\mathbf{U}}\}+[\mathbf{K}]\{\mathbf{U}\}=\{\mathbf{R}(t)\}$$

$$[\mathbf{C}]=\alpha[\mathbf{M}]+\beta[\mathbf{K}]\;(proportional\;\; damping)$$

Difficulties:

- **computational models of very large dimensionality** (the smallest 2D problems of any practical value require to use models consisting of 10^6-10^7 elements);
- **very large number of time integration steps** (inversely proportional to the linear dimension of elements);
- **adequacy of continua-based models to reality**

Increase of the element size

(and simultaneously of the time integration step)

+

preserving accuracy

Increase of the convergence
rate of the model

⇩

Increase of computational efficiency

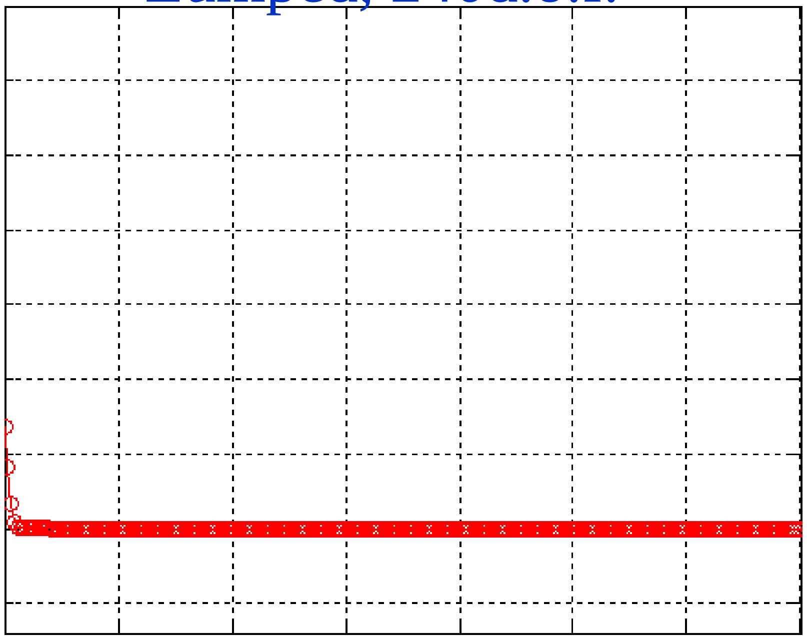
Mass matrices of an element:

- Lumped (diagonal); $\left[\mathbf{M}_L^e\right] = diag(m_i)$
- Consistent; $\left[\mathbf{M}_C^e\right] = \int_V \rho [\mathbf{N}]^T [\mathbf{N}] dV$
- Generalized; $\left[\mathbf{M}_G^e\right] = \alpha \left[\mathbf{M}_C^e\right] + (1-\alpha)\left[\mathbf{M}_L^e\right]$
- ?

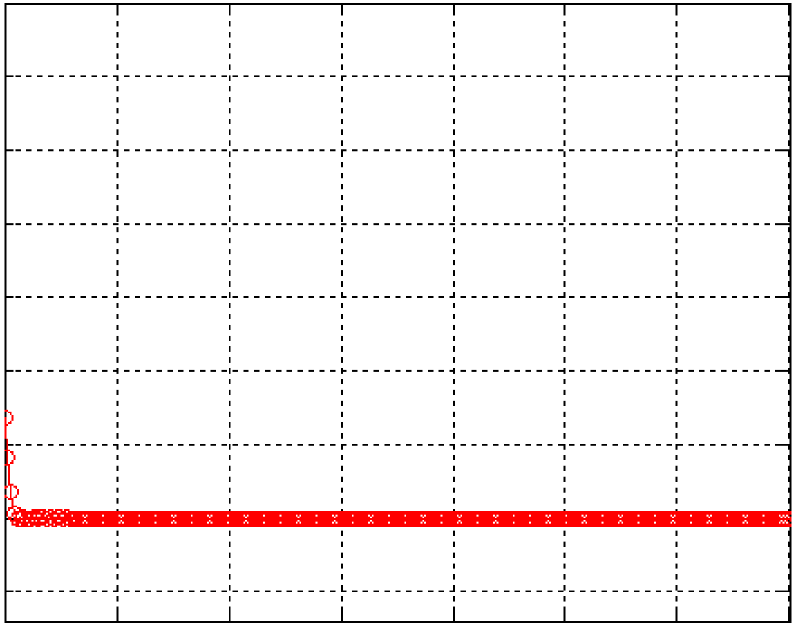
Mass matrices are not uniquely defined and an “optimum” form of them can be found

Wave pulse propagation through discrete models

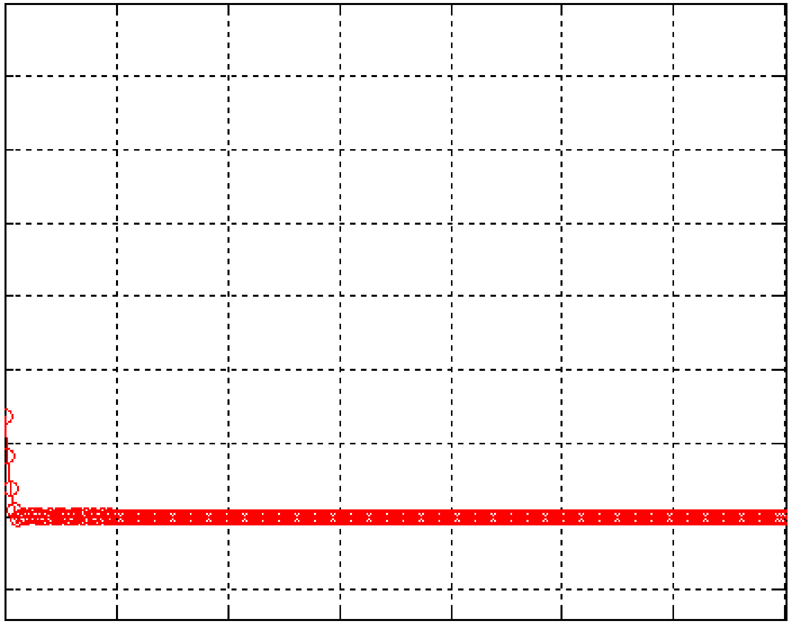
Lumped, 240d.o.f.



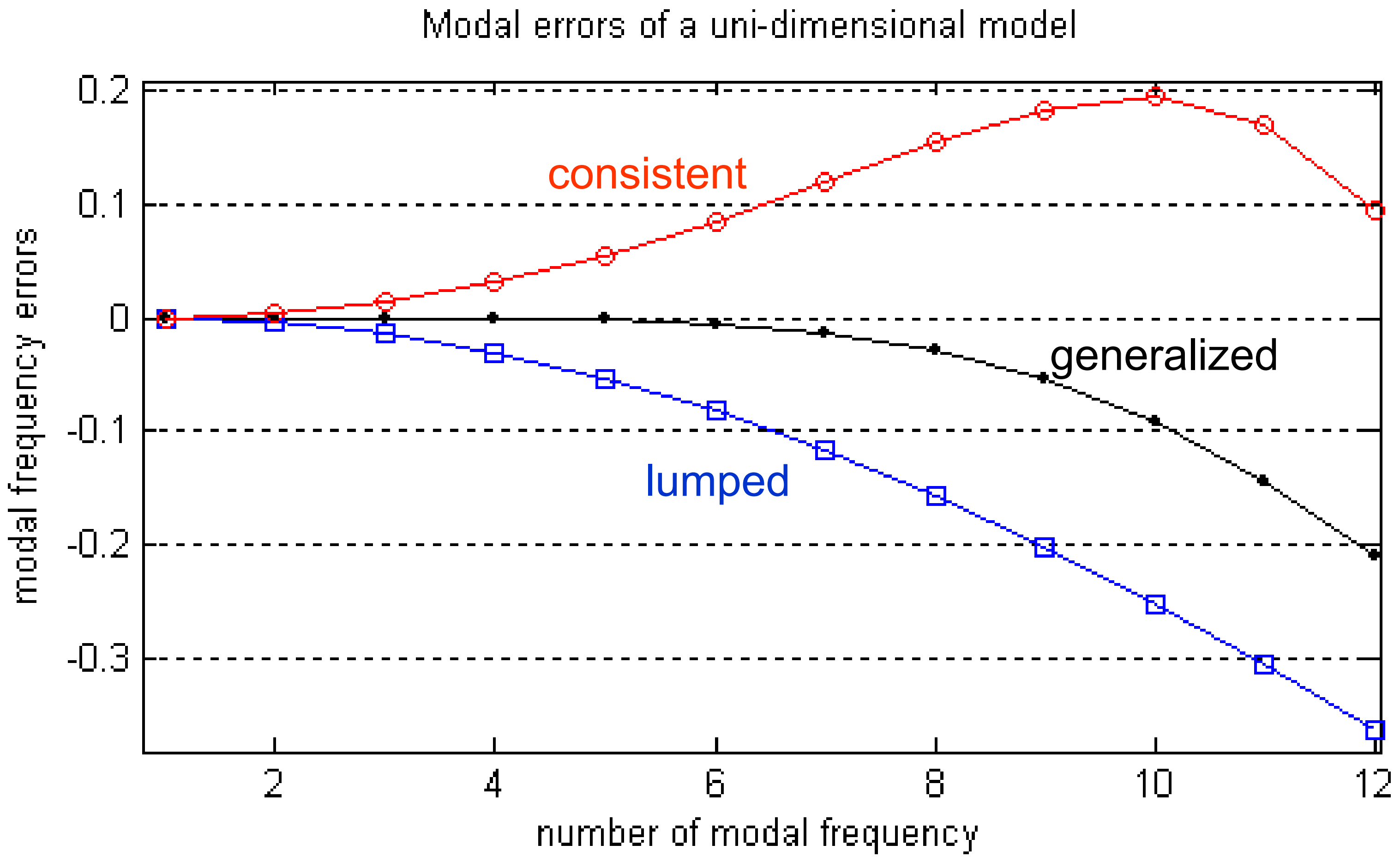
Generalized, 240d.o.f.



Consistent, 240d.o.f.



Modal errors of a unidimensional model (12 nodes)



- The numerical distortion of a wave propagating in a structure is determined mainly by *modal frequency errors of the model*

$$\det\left(\left[\mathbf{K}\right]-\omega^2\left[\mathbf{M}\right]\right)=0$$

$$\omega_1,\omega_2,\ldots,\omega_N$$

modal frequencies

$$\left[\mathbf{Y}\right]=\left[\{\mathbf{y}_1\},\{\mathbf{y}_2\},\ldots,\{\mathbf{y}_N\}\right]$$

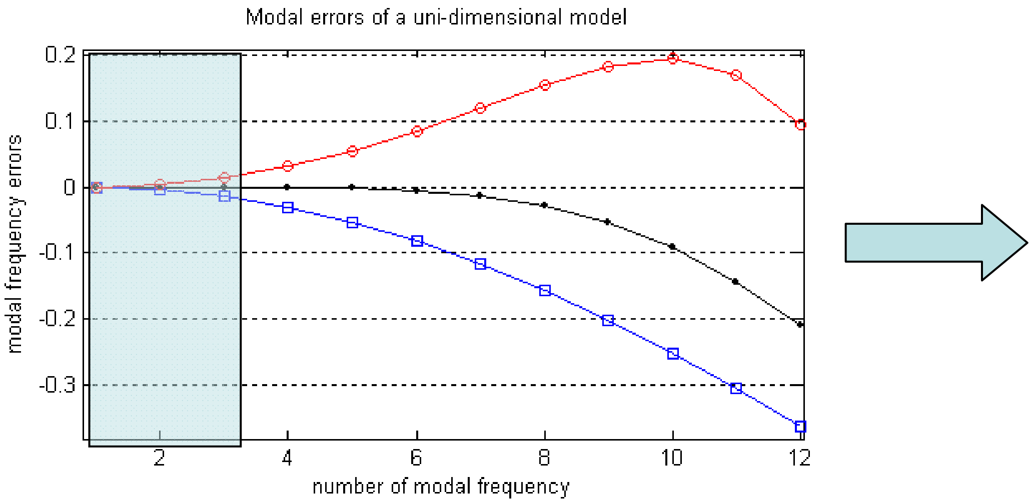
modal shapes

Modal frequency errors cause different propagation velocities of harmonic components *(phase errors)*. The amount of distortion of the wave shape increases with the time of propagation.

- **Real computational models *<u>always(?)</u>* have modal errors**

- **Practically the influence of modal errors is reduced by using a *highly refined mesh***

- **The mesh density must provide that lower 'close to exact' *modal frequencies cover the frequency range of meaningful harmonic components* of the propagating wave**



Practical recommendations:

17 (30-40?) nodes per sine wave length

Definition of an 'ideal' discrete model

- **An** ***'ideal' discrete model*** **of wave propagation in a closed domain represents the** ***modal frequencies of all modes equal to exact modal frequencies of the continuous domain*** **of the same shape.**

How to build a model ensuring 'close to exact' modal frequencies and shapes over all modal frequency range?

Modal synthesis relations

$$\left[\tilde{\mathbf{M}}\right] = \left(\left[\tilde{\mathbf{Y}}\right]^T\right)^{-1}\left[\tilde{\mathbf{Y}}\right]^{-1};$$

$$\left[\tilde{\mathbf{K}}\right] = \left(\left[\tilde{\mathbf{Y}}\right]^T\right)^{-1}\left[diag(\omega_1^2, \omega_2^2, \ldots, \omega_n^2)\right]\left[\tilde{\mathbf{Y}}\right]^{-1}$$

$\left[\mathbf{Y}\right], \omega_1, \omega_2, \ldots, \omega_n$

'Close-to-exact' modal shapes and frequencies known analytically or obtained in a finely discretized model

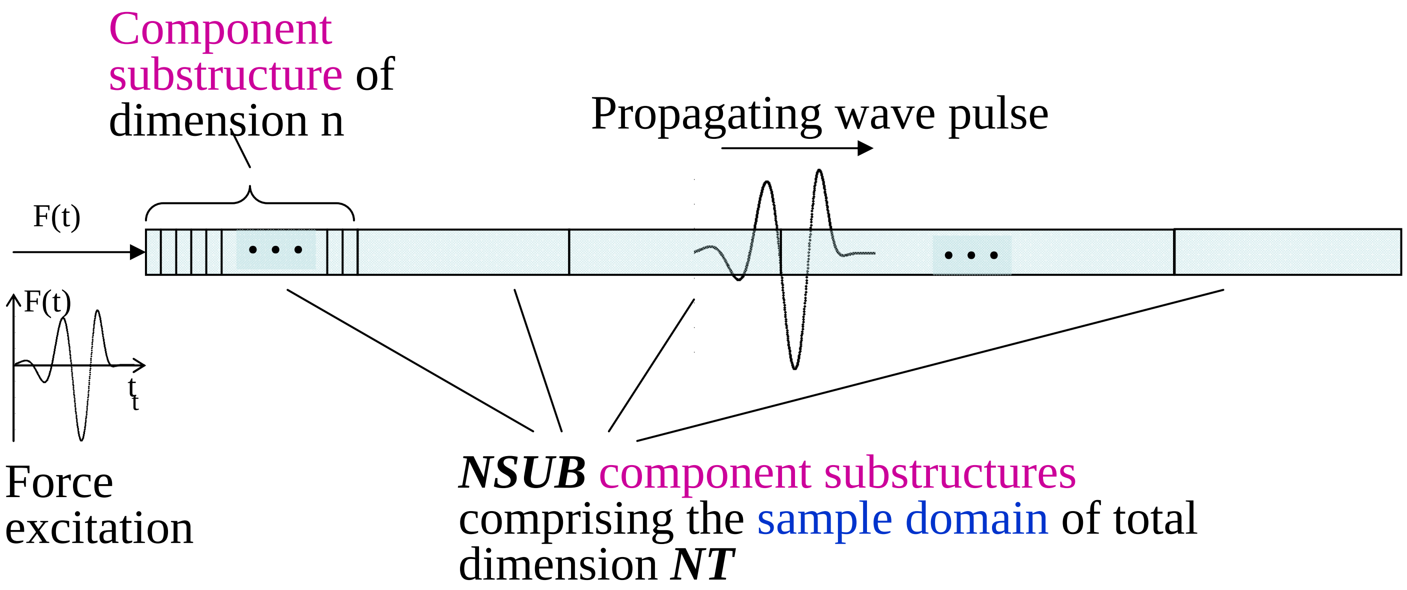
$\left[\tilde{\mathbf{Y}}\right]$ - Modal shapes $\left[\mathbf{Y}\right]$ approximated in a rough mesh

Modal synthesis practically cannot be applied for real computational domains:

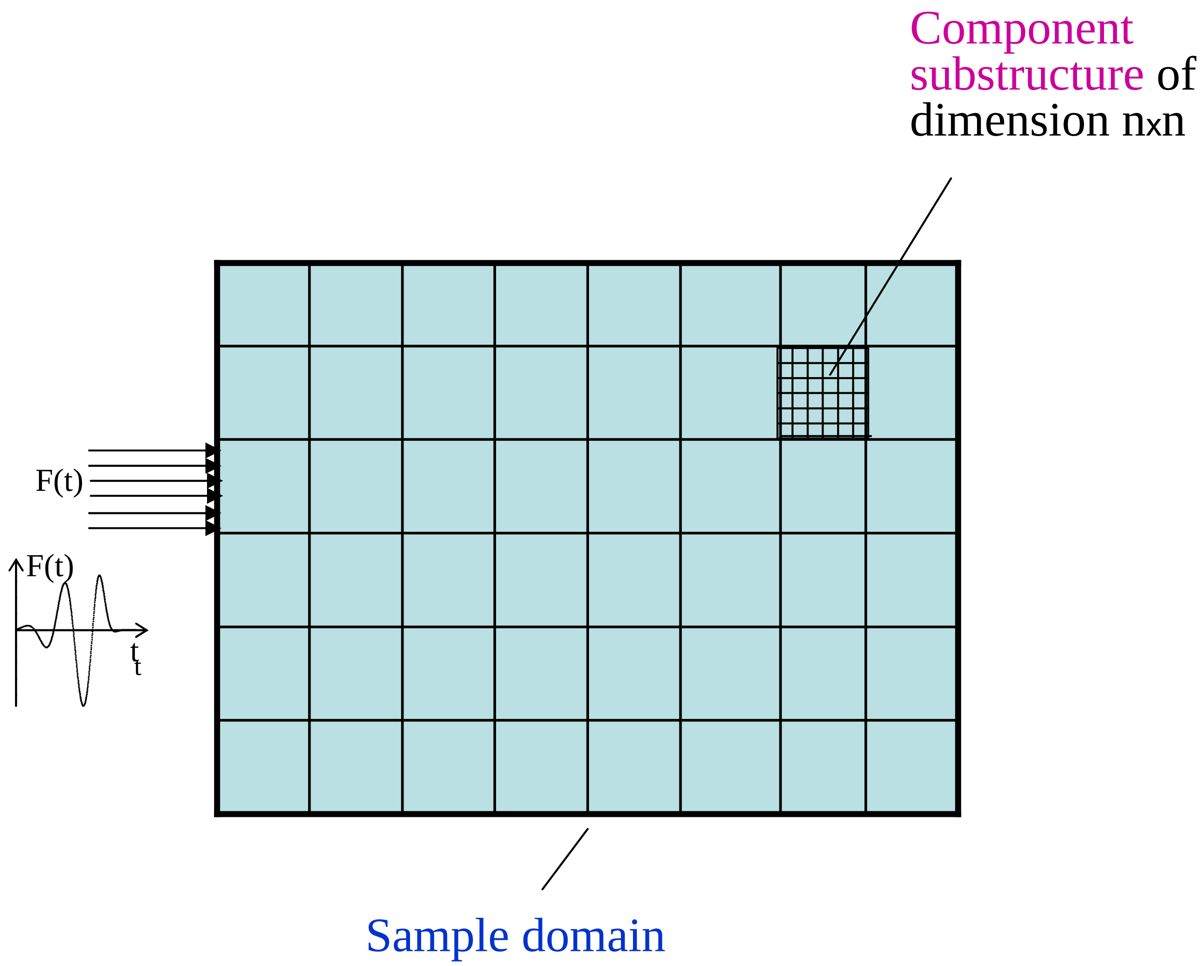
- They are too large to compute the matrix inverse $\left[\tilde{\mathbf{Y}}\right]^{-1}$

- we <u>*do not know*</u> exact modes of the domain

1D Component substructures



2D Component substructures



Problem formulation

- Stand-alone *component substructures* obtained by synthesizing ‘exact’ modes are ‘ideal’;
- Unfortunately, structures assembled of ‘ideal’ component substructures always ***produce significant modal errors.***

Problem: ***Obtain the matrices of a component substructure(CS) such that structure of any size and shape formed by assembling together the matrices of CS would have as many as possible close-to-exact values of modal frequencies***

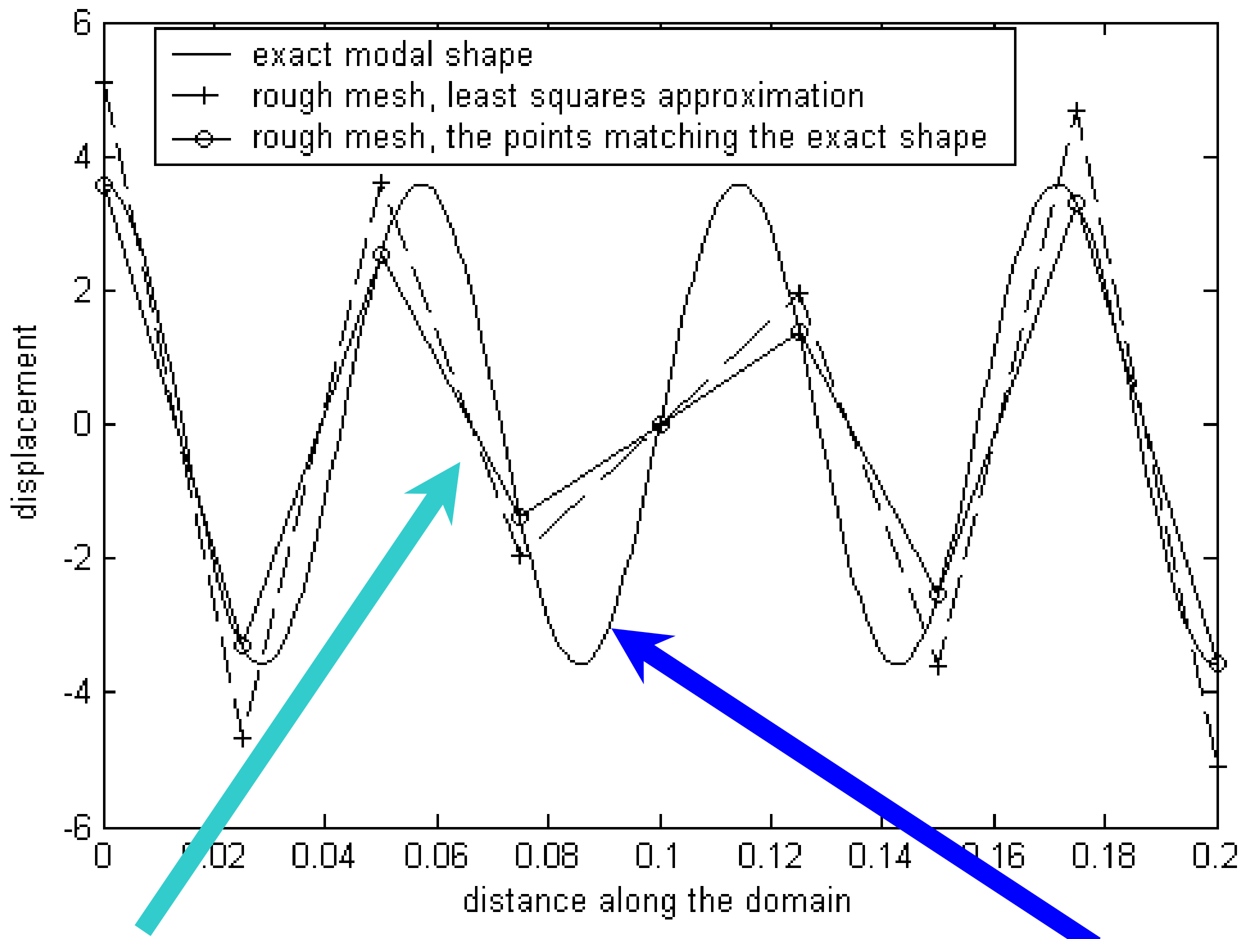
Way of solution

•Matrices of a *component substructure(CS)* are optimized in order to provide the best spectral properties to a *sample domain* assembled of CS;

•Optimized CS can be used to form larger *computational domains* as higher-order elements (superelements);

•Computational domains assembled of optimized CS preserve approximately the same percentage of close-to-exact modal frequencies as was obtained for a sample domain

Non-uniqueness of approximation of ‘exact’ modal shapes in rough meshes



Modal shapes presented in a rough mesh: Least squares or point-to-curve approximation

Exact (theoretical) modal shape

Optimum spectral properties of a component substructure

Find optimum *modification of spectral properties of a component substructure* in order to produce minimum modal error of sample computational domains

$$\left[diag\left(0,\ldots,0,\ \alpha_{r+1}^{\omega}\omega_{r+1}^{2},\ \alpha_{r+2}^{\omega}\omega_{r+2}^{2},\ldots,\ \alpha_{r+n}^{\omega}\omega_{n}^{2}\right)\right] = \left[diag\left(\omega^{2}\right)\right]\left\{\boldsymbol{\alpha}^{\omega}\right\}$$

$$\left[\left\{\tilde{\mathbf{y}}_1\right\},\ldots,\left\{\tilde{\mathbf{y}}_r\right\},\alpha_{r+1}^{y}\left\{\tilde{\mathbf{y}}_{r+1}\right\},\ldots,\alpha_{n}^{y}\left\{\tilde{\mathbf{y}}_{n}\right\}\right] = \left[\tilde{\mathbf{Y}}\right]\left\{\boldsymbol{\alpha}^{y}\right\}$$

$$\left[\tilde{\mathbf{y}}\ \right] = \beta_i^l\left[\tilde{\mathbf{y}}_{\delta}\right] + \left(1-\beta_i^l\right)\left[\tilde{\mathbf{y}}_{c}\right], \qquad 0 < \beta_i^l < 1$$

Find the values of coefficients $\left\{\boldsymbol{\alpha}^{\omega}\right\}$, $\left\{\boldsymbol{\alpha}^{y}\right\}$ ***and*** $\beta_i^l,\ i = 1,\ldots,n$

Minimization problem

$$\min_{\{\boldsymbol{\alpha}^{\omega}\},\{\boldsymbol{\alpha}^{y}\},\beta_k^l} \Psi$$

$$\Psi = \sum_{i=r+1}^{\hat{N}} \left(\frac{\hat{\omega}_i - \hat{\omega}_{i0}}{\hat{\omega}_{i0}} \right)^2$$ - target function;

$\hat{\omega}_i$ - modal frequency of *i*-th mode of the ***sample domain*** *obtained by using the current set of optimization variables;*

$\hat{\omega}_{i0}$ - exact value of the modal frequency of *i*-th mode of the ***sample domain*** *;*

The geometrical shape of the sample domain is selected freely by assembling together several CS

Gradients of the target function

$$\frac{\partial \Psi}{\partial \alpha_j^y} = \sum_{i=1}^{\hat{N}} \frac{\hat{\omega}_i - \hat{\omega}_{i0}}{\hat{\omega}_{i0}^2 \hat{\omega}_i} \{\hat{\mathbf{y}}_i\}^T \left(\frac{\partial \left[\hat{\mathbf{K}}\right]}{\partial \alpha_j^y} - \omega_i^2 \frac{\partial \left[\hat{\mathbf{M}}\right]}{\partial \alpha_j^y} \right) \{\hat{\mathbf{y}}_i\},$$

$$\frac{\partial \Psi}{\partial \alpha_j^\omega} = \sum_{i=1}^{\hat{N}} \frac{\hat{\omega}_i - \hat{\omega}_{i0}}{\hat{\omega}_{i0}^2 \hat{\omega}_i} \{\hat{\mathbf{y}}_i\}^T \left(\frac{\partial \left[\hat{\mathbf{K}}\right]}{\partial \alpha_j^\omega} - \omega_i^2 \frac{\partial \left[\hat{\mathbf{M}}\right]}{\partial \alpha_j^\omega} \right) \{\hat{\mathbf{y}}_i\},$$

$$\frac{\partial \Psi}{\partial \beta_j^l} = \sum_{i=1}^{\hat{N}} \frac{\hat{\omega}_i - \hat{\omega}_{i0}}{\hat{\omega}_{i0}^2 \hat{\omega}_i} \{\hat{\mathbf{y}}_i\}^T \left(\frac{\partial \left[\hat{\mathbf{K}}\right]}{\partial \beta_j^l} - \omega_i^2 \frac{\partial \left[\hat{\mathbf{M}}\right]}{\partial \beta_j^l} \right) \{\hat{\mathbf{y}}_i\}$$

Expressions of derivatives:

$$\frac{\partial[\tilde{\mathbf{M}}]}{\partial\alpha_j^y}=-\left(\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T\right)^{-1}\left([0,\dots0,\{\tilde{\mathbf{y}}_j\},0,\dots0]^T[\tilde{\mathbf{M}}][\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}+\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T[\tilde{\mathbf{M}}][0,\dots0,\{\tilde{\mathbf{y}}_j\},0,\dots0]\right)\left([\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}\right)^{-1};$$

$$\frac{\partial[\tilde{\mathbf{K}}]}{\partial\alpha_j^y}=-\left(\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T\right)^{-1}\left([0,\dots0,\{\tilde{\mathbf{y}}_j\},0,\dots0]^T[\tilde{\mathbf{K}}][\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}+\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T[\tilde{\mathbf{K}}][0,\dots0,\{\tilde{\mathbf{y}}_j\},0,\dots0]\right)\left([\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}\right)^{-1};$$

$$\frac{\partial[\tilde{\mathbf{M}}]}{\partial\alpha_j^\omega}=0\ ;$$

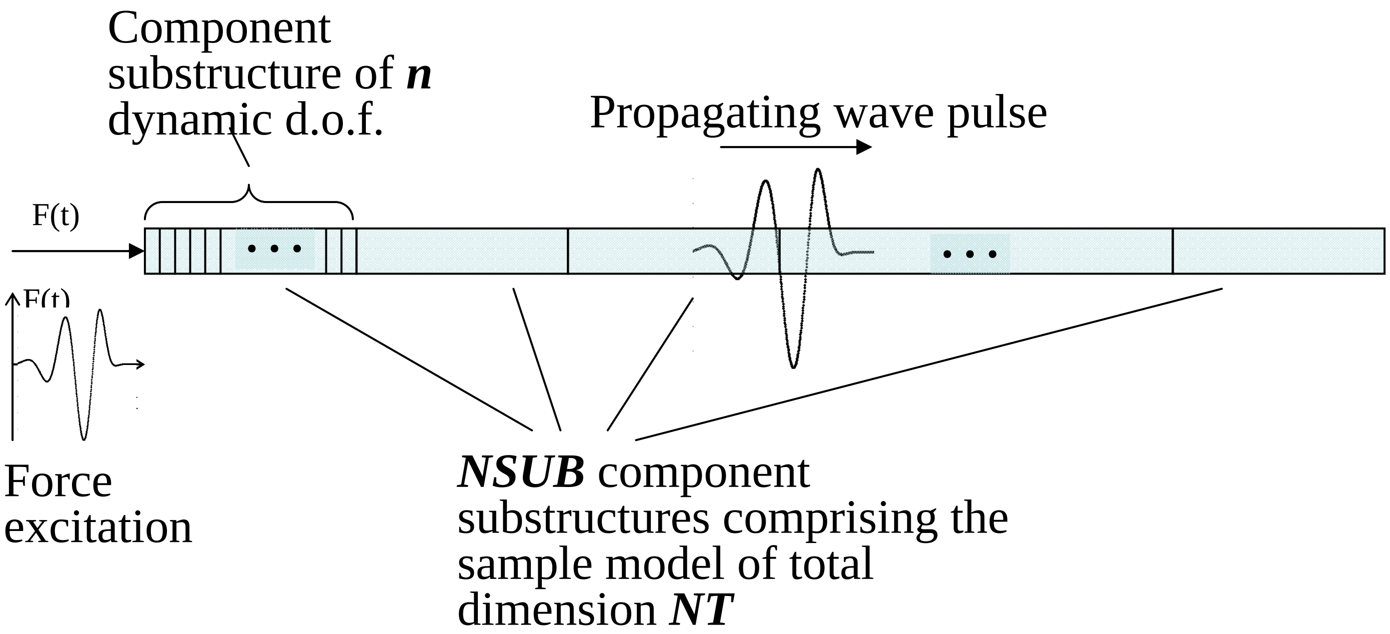
$$\frac{\partial[\tilde{\mathbf{K}}]}{\partial\alpha_j^\omega}=\left(\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T\right)^{-1}\left[diag\left(0,\dots,0,\omega_j^2,0,\dots,0\right)\right]\left([\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}\right)^{-1}$$

$$\frac{\partial[\tilde{\mathbf{M}}]}{\partial\beta_j^l}=-\left(\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T\right)^{-1}\left(\left[0,\dots0,\alpha_j^y\frac{\partial\{\tilde{\mathbf{y}}_j\}}{\partial\beta_j^l},0,\dots0\right]^T[\tilde{\mathbf{M}}][\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}+\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T[\tilde{\mathbf{M}}]\left[0,\dots0,\alpha_j^y\frac{\partial\{\tilde{\mathbf{y}}_j\}}{\partial\beta_j^l},0,\dots0\right]\right)\left([\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}\right)^{-1};$$

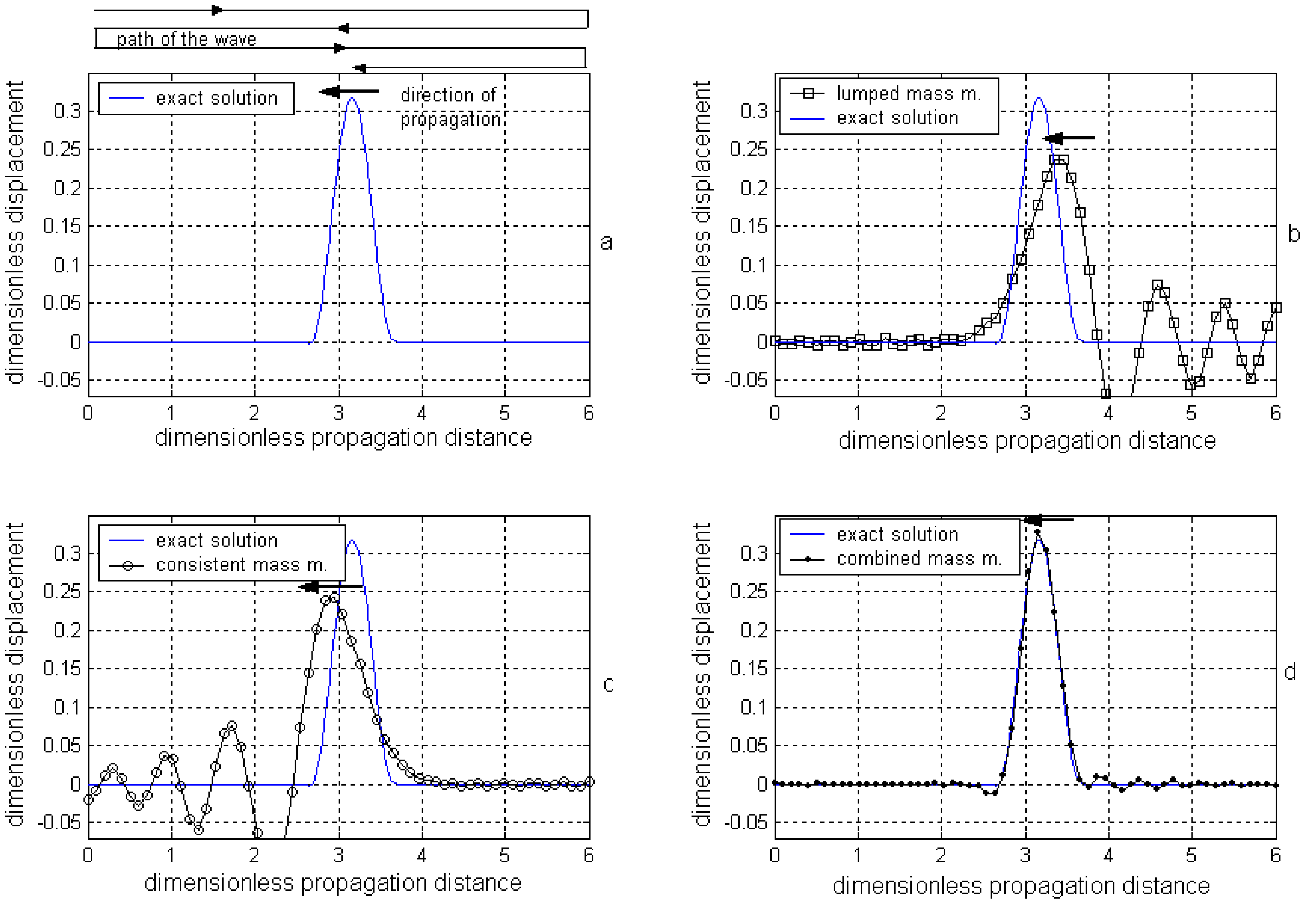
$$\frac{\partial[\tilde{\mathbf{K}}]}{\partial\beta_j^l}=-\left(\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T\right)^{-1}\left(\left[0,\dots0,\alpha_j^y\frac{\partial\{\tilde{\mathbf{y}}_j\}}{\partial\beta_j^l},0,\dots0\right]^T[\tilde{\mathbf{K}}][\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}+\{\boldsymbol{\alpha}^y\}^T[\tilde{\mathbf{Y}}]^T[\tilde{\mathbf{K}}]\left[0,\dots0,\alpha_j^y\frac{\partial\{\tilde{\mathbf{y}}_j\}}{\partial\beta_j^l},0,\dots0\right]\right)\left([\tilde{\mathbf{Y}}]\{\boldsymbol{\alpha}^y\}\right)^{-1};$$

$$j=r+1,\dots,n$$

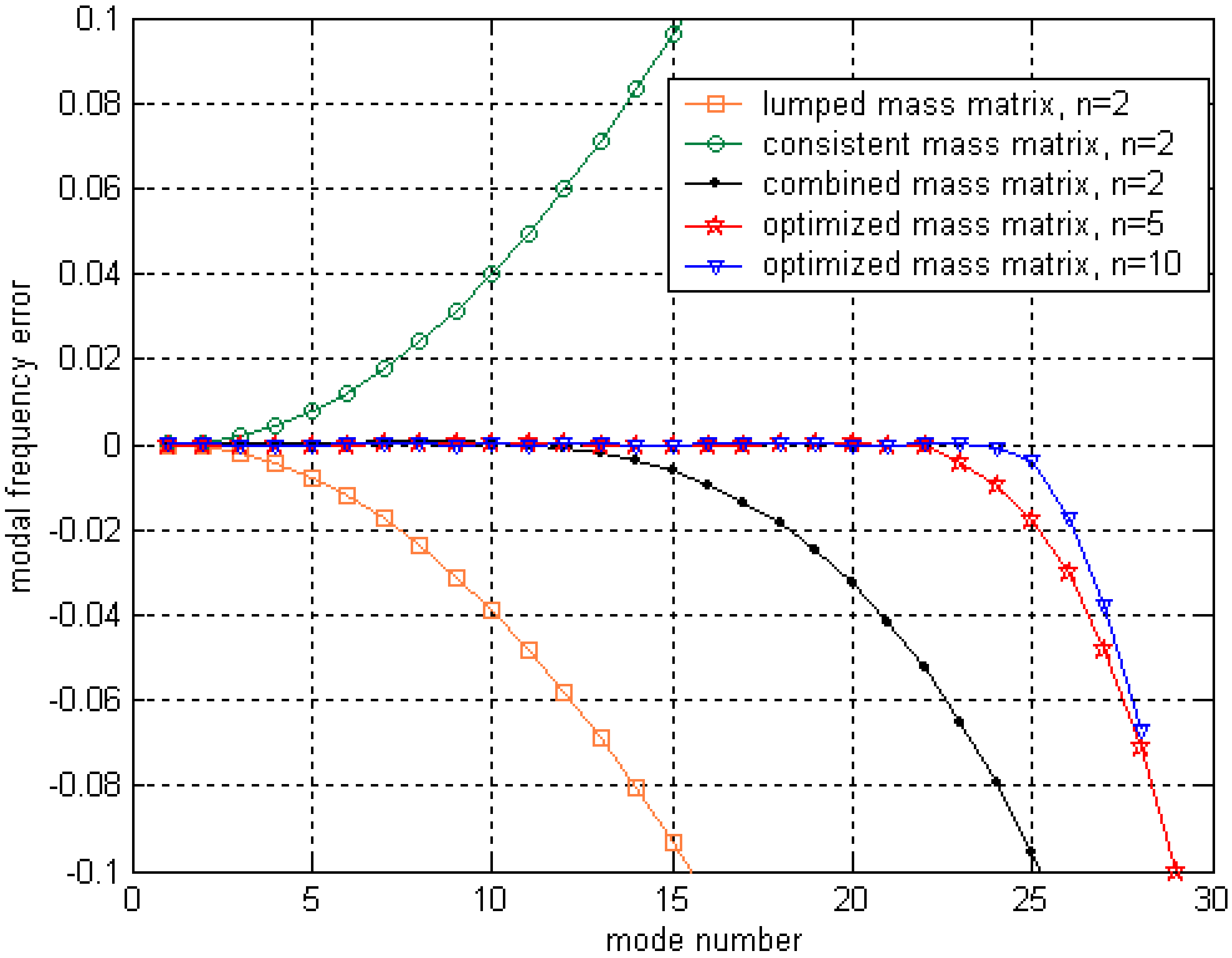
Numerical results in 1D: the waveguide structure



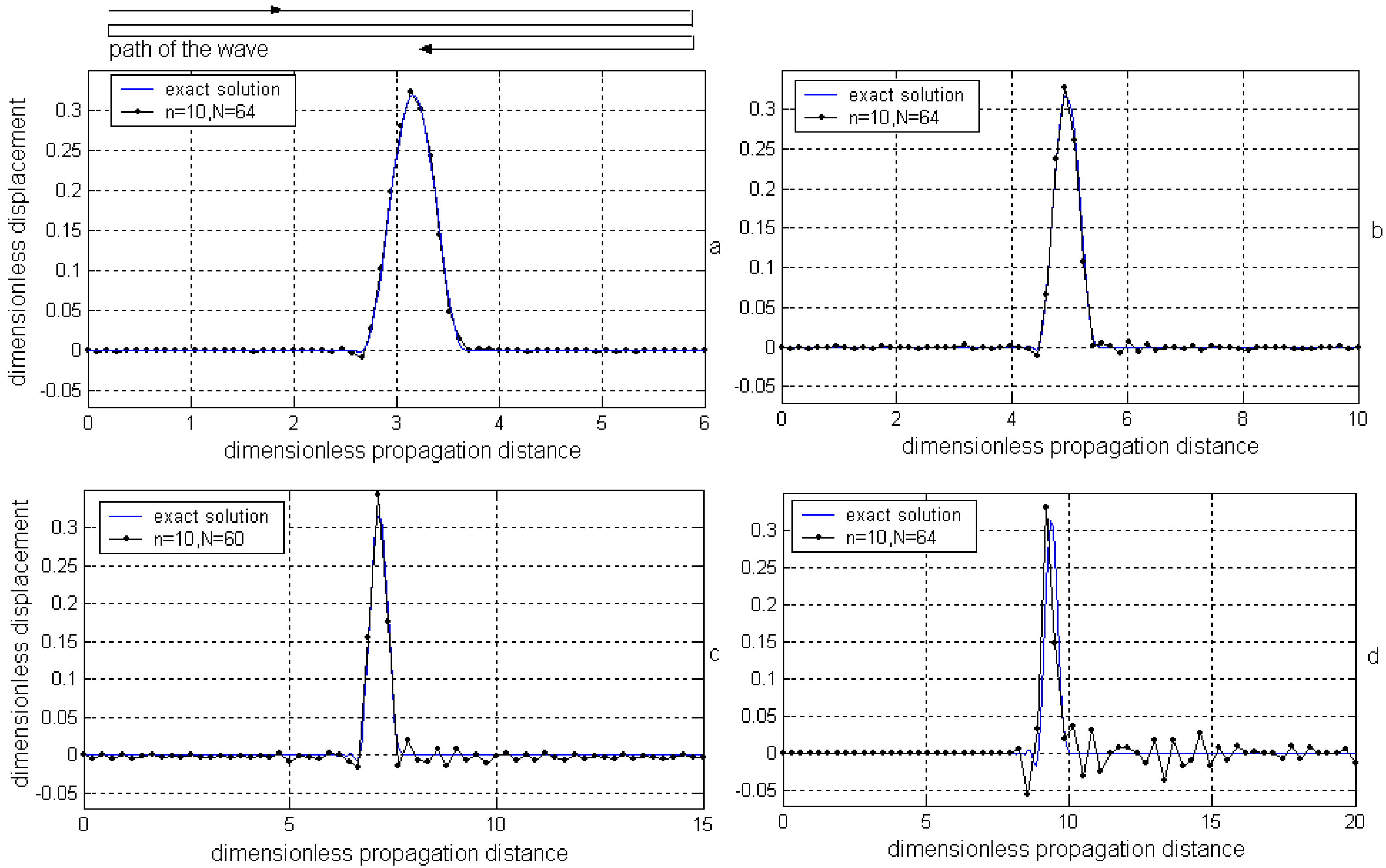
Numerical results: Typical distortions of the shape of a propagating wave pulse in a rough equally spaced mesh



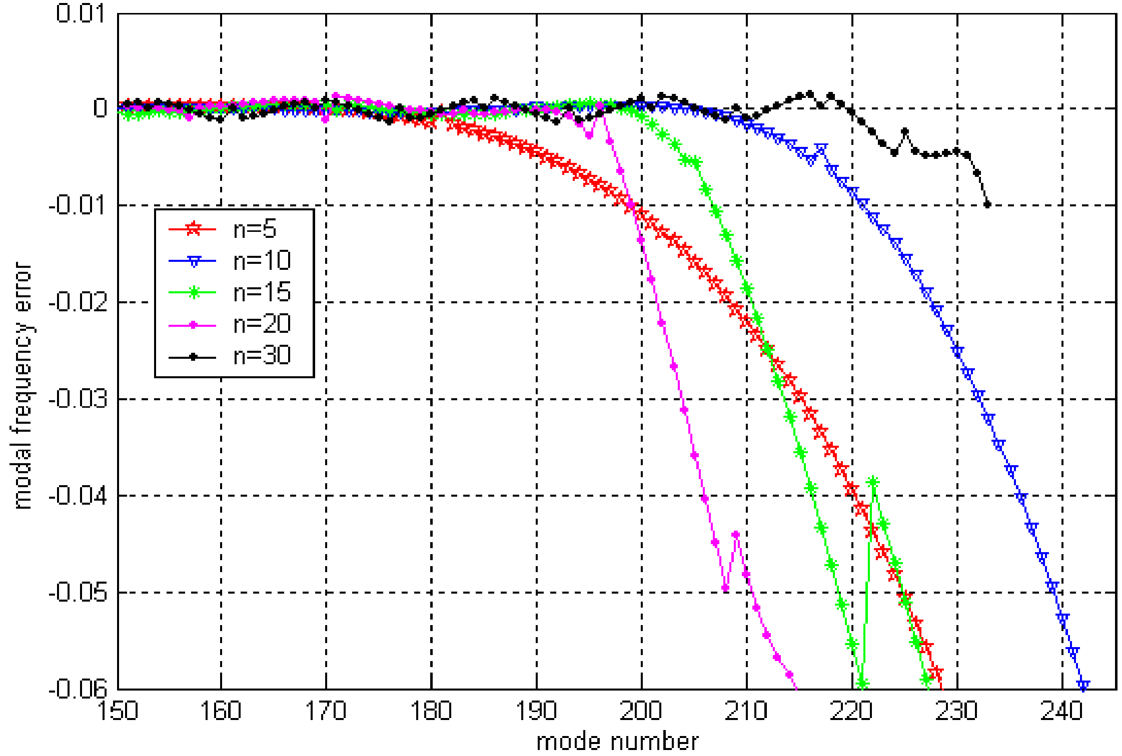
Numerical results: modal frequency errors of a sample domain



Numerical results: shape distortion of a propagating wave pulse in the model assembled of seven 10-node component substructures



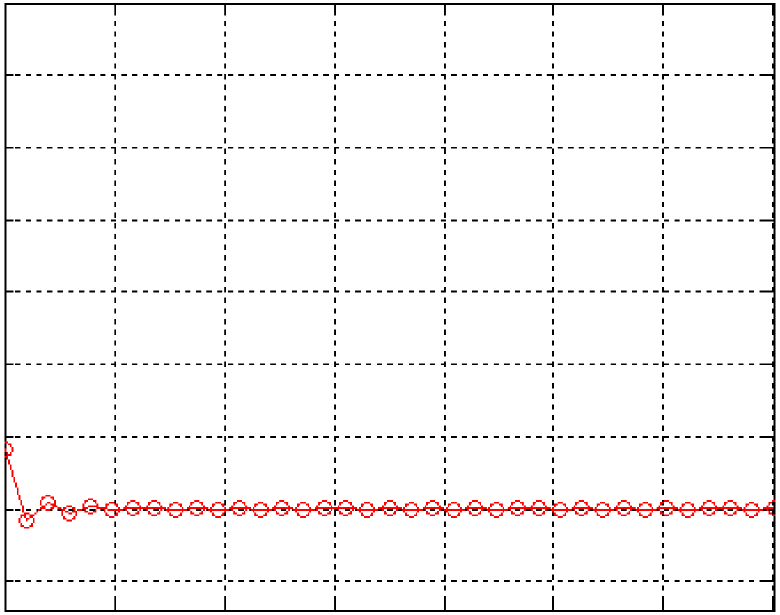
Numerical results: modal frequency errors of a sample domain assembled of optimized CS



Wave pulse propagation through structures models made of synthesized domain models

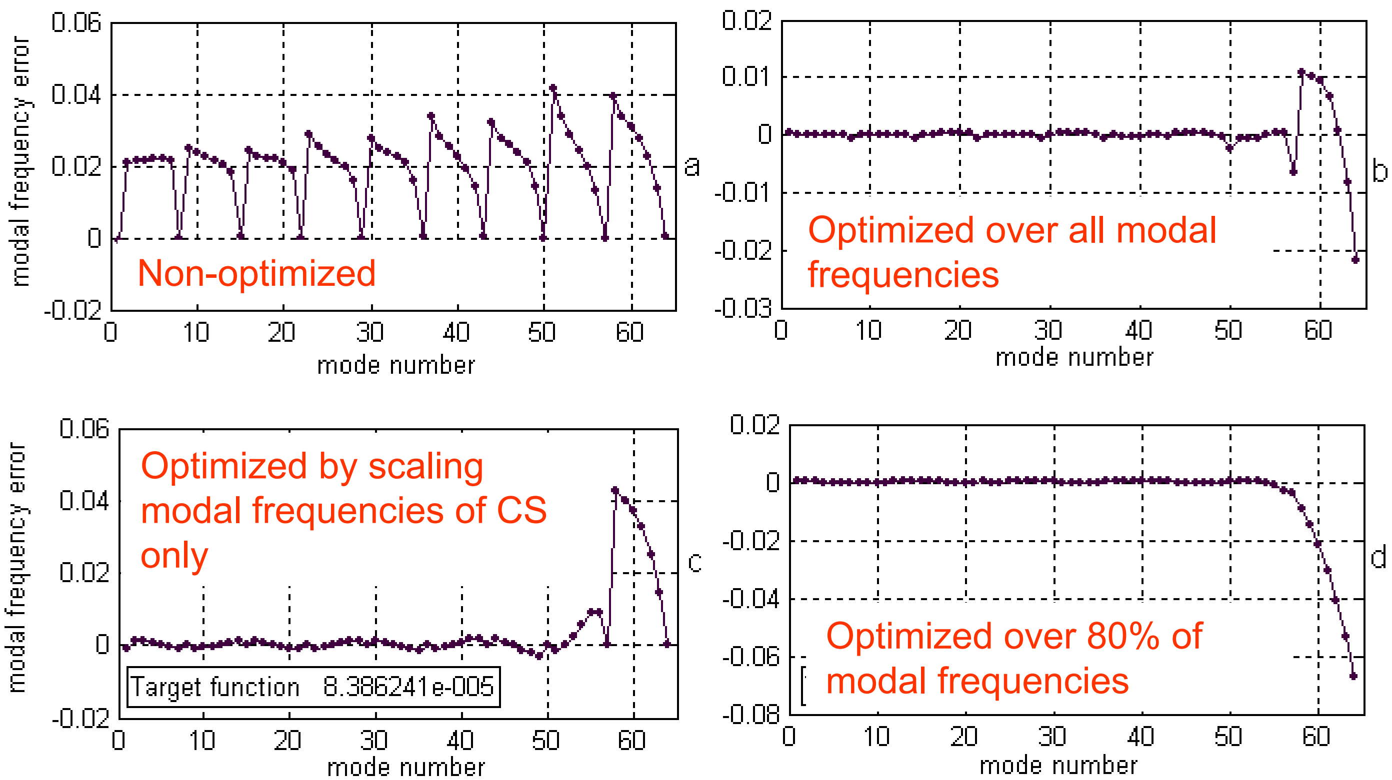
Synthesized_10, 120dof.

Synthesized_10, 40dof.



Synthesized_10, 60dof.

Modal frequency errors of an uni-dimensional waveguide model assembled of 7 CS_10



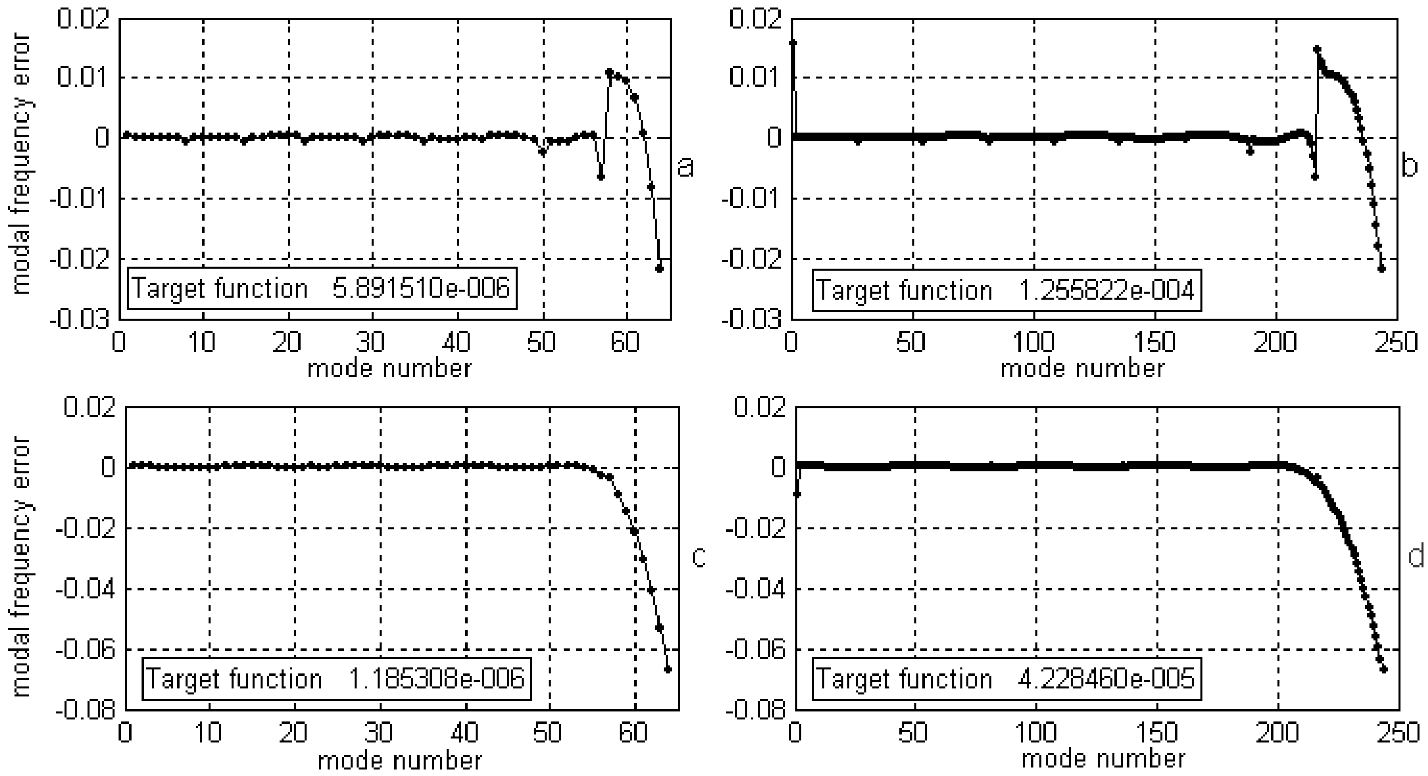
Modal frequency errors of an uni-dimensional waveguide model

Domain assembled of 6 CS_10

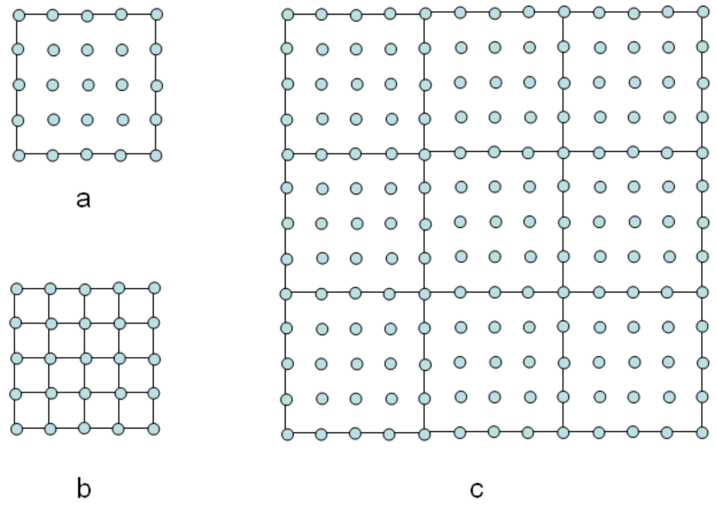
Domain assembled of 24 CS_10

Optimized over all modal frequencies

Optimized over 80% of modal frequencies

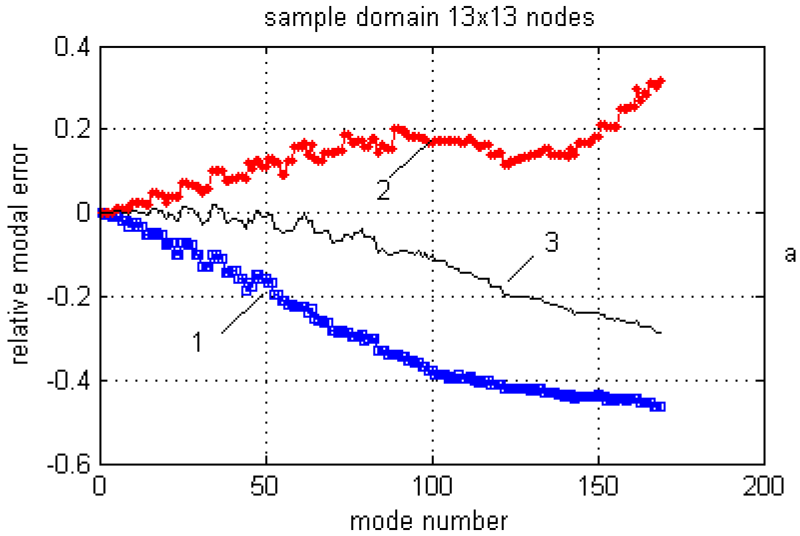


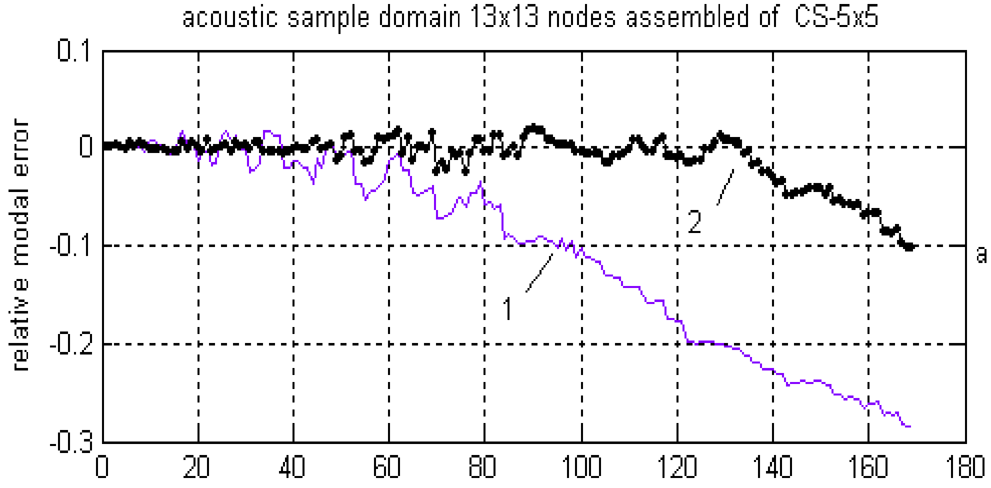
Numerical results in 2D

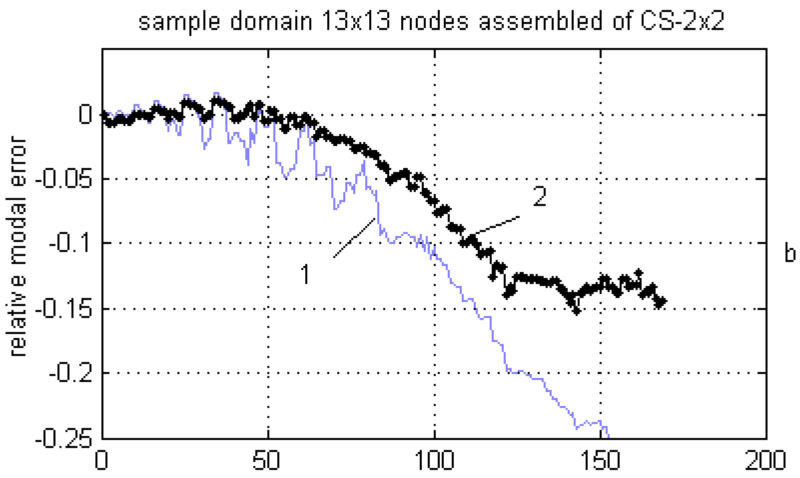


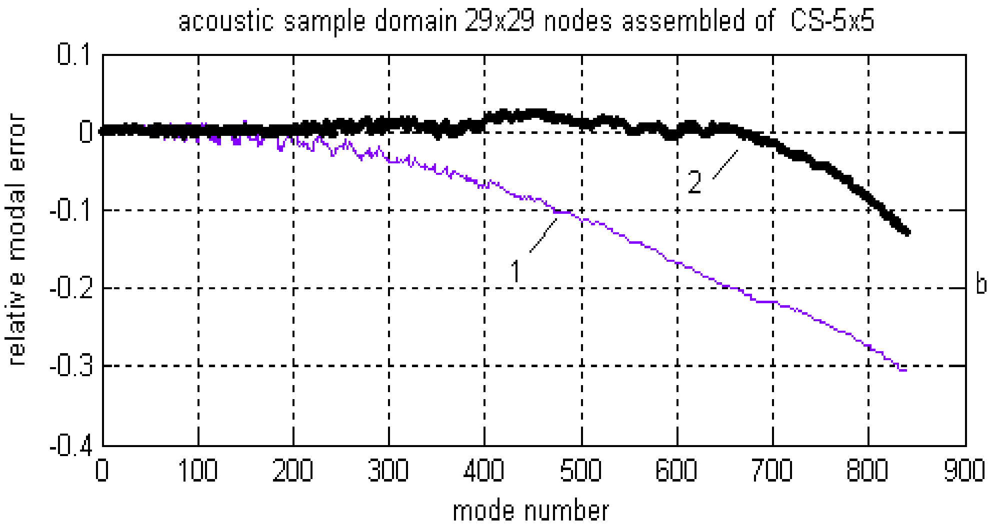
a - component substructure CS_5x5 ;
b - component substructure meshed by quadrilateral elements;
c - quadrilateral sample domain 13x13 nodes assembled of 9 CS_5x5

Numerical results: modal frequency errors of a sample domain

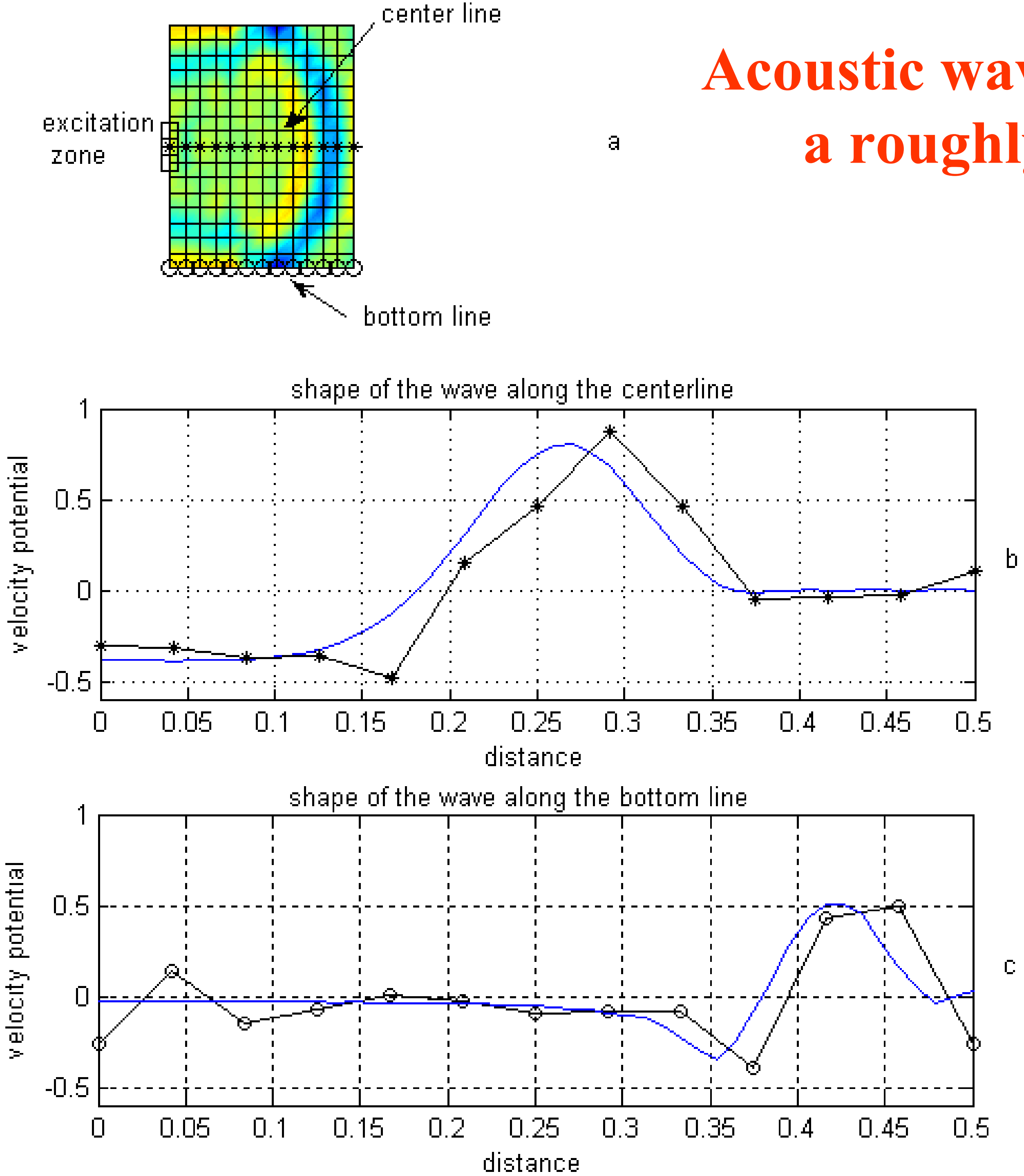




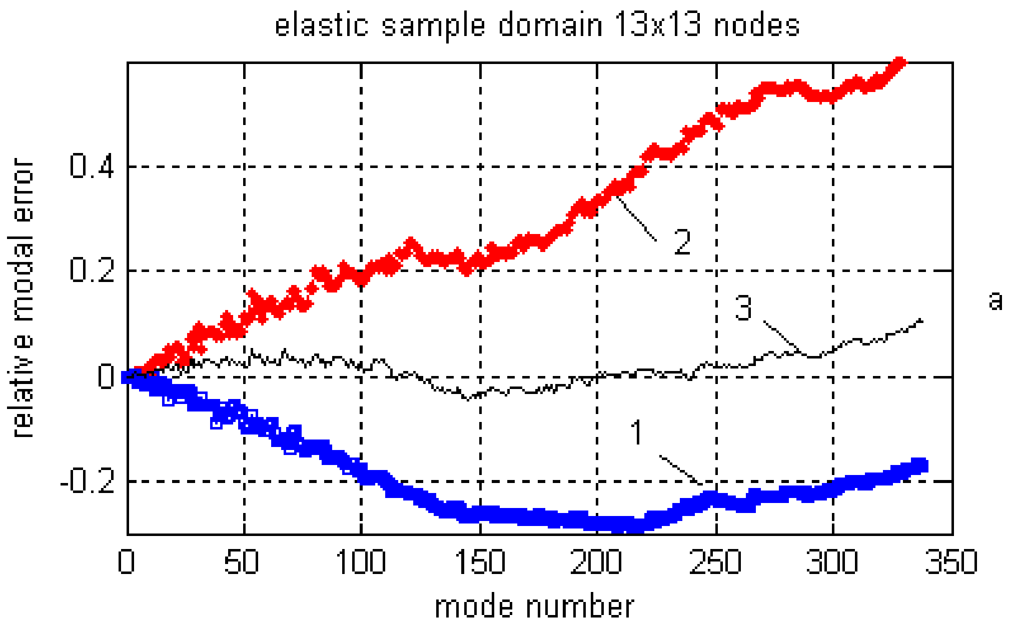


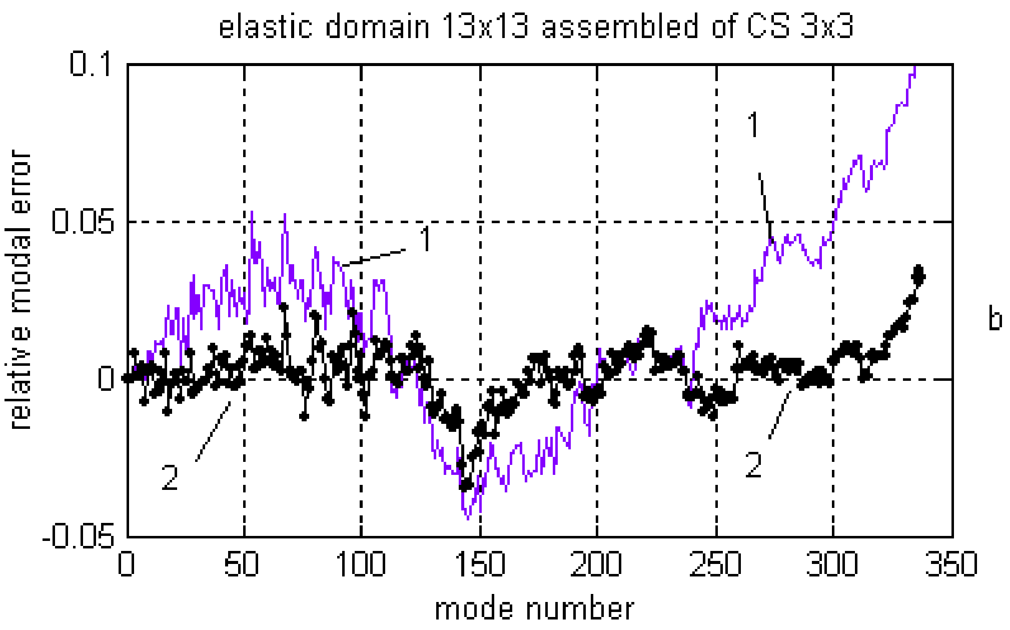


Acoustic wave propagating in a roughly meshed (13x17) domain



Relative modal errors of a quadrilateral elastic sample domain 13x13 nodes





Conclusions -1

- A regular approach has been presented for obtaining the mass and stiffness matrices of component substructures such that after assembling them to a larger model the convergence of modal frequencies is as high as possible;

- Once calculated, the component domain matrices can be used to form any structure and may be interpreted as higher-order elements or super-elements;

Conclusions -2

- The models able to present very close-to-exact modal frequency values of more than ~80% of the total modal frequency number can be obtained;

- The models can be used for modelling short transient wave pulses propagating in elastic or acoustic environments. The distinguishing feature is the <u>ability to present the wave pulse by using very few nodal points per wavelength</u>

- A limitation of the approach is that it works for regular shape domains and elements only. However, freely meshed domains can be used in the same model by using *domain decomposition* techniques